EXAMEN MICRO-200 de Janvier 2024

QUESTION 1 – Poutre Composite *(36 points)*

Une poutre composite de longueur $\boldsymbol{L}$ et de masse négligeable est supportée en A et en B, voir la **Figure 1**, qui est dessinée dans le plan xy. La poutre est soumise à une force distribuée d'intensité q [N/m] et à une force ponctuelle $P = qL/2$.

La poutre est composée de deux matériaux. Le matériau en vert a un module de Young $\boldsymbol{E_1}$. Le matériau en bleu a un module de Young $\boldsymbol{E_2 = 2E_1}$. La section a une largeur b et une hauteur $2a$.

Pour les questions Q1b et Q1c, la section de la poutre est illustrée en **Figure 2**, dans le plan yz. Il y a un trou (vide) de largeur $b/2$ et de hauteur d.

Pour les questions Q1d à Q1g, la section de la poutre est indiquée en **Figure 3,** dans le plan yz.

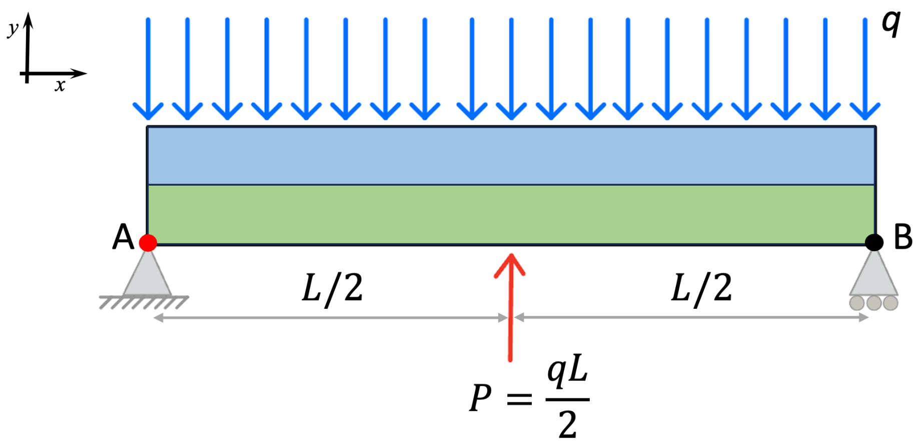


Figure 1: Poutre AB sujet à une force distribuée q et à une force ponctuelle P. Le système de coordonnées xy a le point A comme origine.

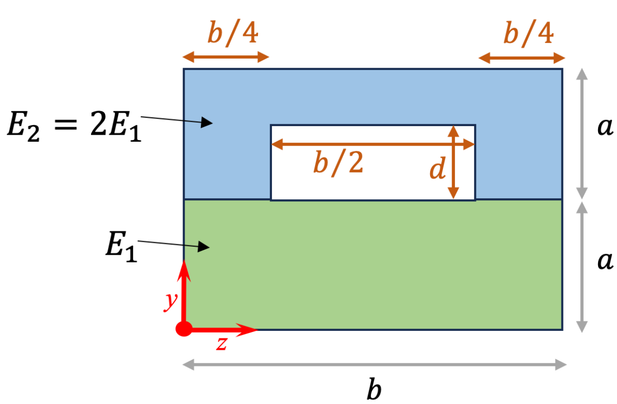


Figure 2: Section de la poutre composite pour les questions Q1b et Q1c.
L'origine du système de coordonnées yz est indiquée.

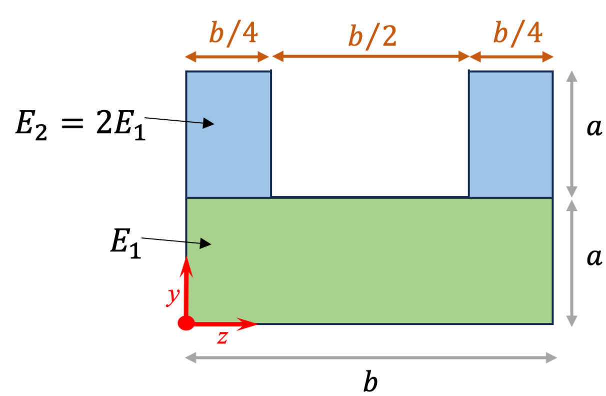


Figure 3: Section de la poutre composite pour les questions Q1d à Q1g.
L'origine du système de coordonnées yz est indiquée.

- **Q1a)** (4 pts)
  - Dessinez le diagramme des forces de la poutre AB (avec un système de coordonnées).
  - Calculez toutes les forces et tous les moments de réaction sur la poutre.
  - Est-ce que la poutre est statiquement indéterminée ?

Pour les deux questions suivantes (Q1b et Q1c), la section de la poutre est donnée en Figure 2.

- **Q1b)** (7 pts)
  - Calculez la position de l'axe neutre y_0 en fonction de a, b, et d dans le système de coordonnées indiqué en Figure 2 (c'est-à-dire avec $y = 0$ en bas de la poutre).
- **Q1c)** (2 pts)
  - Quelle doit être la valeur de d pour que $y_0 = a$? Justifiez votre réponse

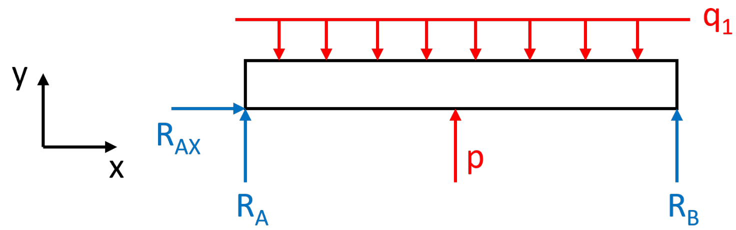
Pour les questions suivantes (Q1d à Q1g), la section de la poutre a changé et est donnée en Figure 3 (dans le plan yz).

- **Q1d)** (6 pts)
  - Calculer la rigidité en flexion $\langle EI_{z,y_0} \rangle$ de la poutre en Figure 3 en fonction seulement de a, b, et E_1.
    - *Indice* : vous pouvez utiliser que $y_0 = a$
- **Q1e)** (5 pts)
  - Calculer la flèche $w(x)$ le long de la poutre en fonction seulement de x, q, L, et $\langle EI_{z,y_0} \rangle$.
    - *Indice1:* Vous pouvez utiliser les formules dans le tableau annexé au formulaire.
    - *Indice2:* Vous pouvez utiliser le changement de variable $x \rightarrow (L - x)$ si le tableau ne donne la flèche que pour x entre 0 et $L/2$.
- **Q1f)** (7 pts)
  - Trouvez le moment de flexion interne $\boldsymbol{M_Z(x)}$ le long de la poutre en fonction de: x, q et L.
  - Pour quelle valeur (ou valeurs) de x est-ce que $|\boldsymbol{M_Z(x)}|$ est maximum ? (Justifier)
  - Quelle est la valeur de $\boldsymbol{M_Z(x)}$ en ce point, en fonction de q et de L?
    - Indice : vous êtes libre d'utiliser la méthode de votre choix pour trouver $\boldsymbol{M_z(x)}$, mais la méthode des sections nous semble la plus simple pour ce problème.
- **Q1g)** (5 pts)
  - à x = L/4, calculez la contrainte $\boldsymbol{\sigma_X(y)}$. Donnez votre réponse en fonction de: { y, E_1, $\langle EI_{z,y_0} \rangle$, L, et q } ou de { y, E_1, $\langle EI_{z,y_0} \rangle$, et $M_z(x = \frac{L}{4})$}. Travaillez dans le système de coordonnées de la Figure 3.
  - Dessinez $\boldsymbol{\sigma_X(y)}$ en fonction de y.
  - Pour quelle valeur de y est-ce que $|\sigma_X(y)|$ est maximum ? Cette contrainte maximale (en valeur absolue) est-elle en traction ou en compression ? Justifier.

Solution

Q1a)

Free body diagram



Sum of forces along x:

$$\sum F_x = R_{AX} = 0$$

$$R_{AX} = 0$$

Sum of forces along y:

$$\sum F_y = Lq_1 - P - R_A - R_B = 0$$

$$R_A + R_B = \frac{1}{2} L q_1$$

Sum of moments at point A:

$$\sum M_A = R_B L - q_1 L \frac{L}{2} + P \frac{L}{2} = 0$$

By substituting P,

$$R_B L - q_1 \frac{L^2}{2} + q_1 \frac{L^2}{4} = 0$$

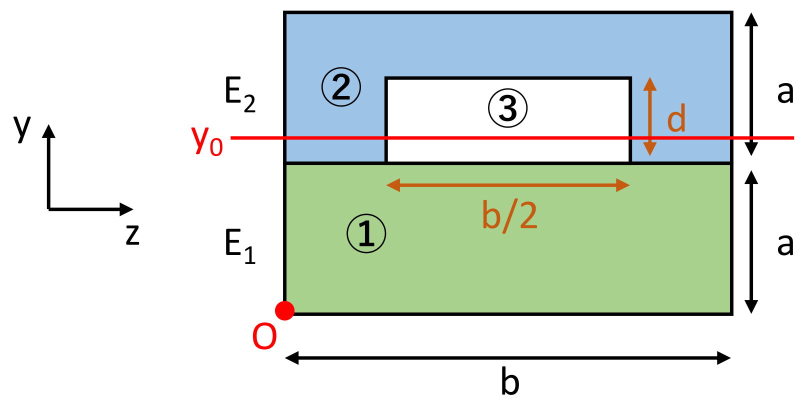
$$R_B = q_1 \frac{L}{4}$$

$$R_A = q_1 \frac{L}{4}$$

All unknown reactive forces can be determined from the equations of equilibrium alone, so the beam is statically determinate.

Q1b)

We will divide the beam into 3 zones (① - ③) shown in the following diagram. Zone 3 is a hole. Zone 2 is a full rectangle of material E_2. Zone 1 is a full rectangle of material E_1.



The position of the neutral axis y_0 can be calculated by

$$y_0 = \frac{\int_A E(y)y\,dydz}{\int_A E(y)dydz}$$
$$= \frac{T_1 + T_2 - T_3}{A_1 + A_2 - A_3}$$

As stated in the question, we set y = 0 at the bottom of the beam.

Zone 1:

$$A_1 = \int_1 E_1 dydz = \int_{z=0}^{b} E_1 \int_{y=0}^{a} dy\,dz = E_1 ab$$
$$T_1 = \int_1 E_1 ydydz = E_1 \int_{z=0}^{b} \int_{y=0}^{a} ydy\,dz = \frac{1}{2} E_1 a^2 b$$

Zone 2:

$$A_2 = E_2 \int_{z=0}^{b} \int_{y=a}^{2a} dy\,dz = 2E_1 ab$$
$$T_2 = E_2 \int_{z=0}^{b} \int_{y=a}^{2a} ydy\,dz = 2E_1 b \frac{1}{2} y^2 \Big|_a^{2a} = E_1 b(4a^2 - a^2) = 3E_1 a^2 b$$

Zone 3:

$$A_3 = E_2 \int_{b/4}^{3b/4} dz \int_{y=a}^{a+d} dy = 2E_1 \frac{b}{2} d = E_1 bd$$
$$T_3 = E_2 \int_{z=b/4}^{3b/4} dz \int_{y=a}^{a+d} ydy = 2E_1 \frac{b}{2} \frac{y^2}{2}\Big|_a^{a+d} = E_1 \frac{b}{2}[(a+d)^2 - a^2] = E_1 \frac{b}{2}(2ad + d^2)$$

We can now combine, remembering to subtract the hole and add the material:

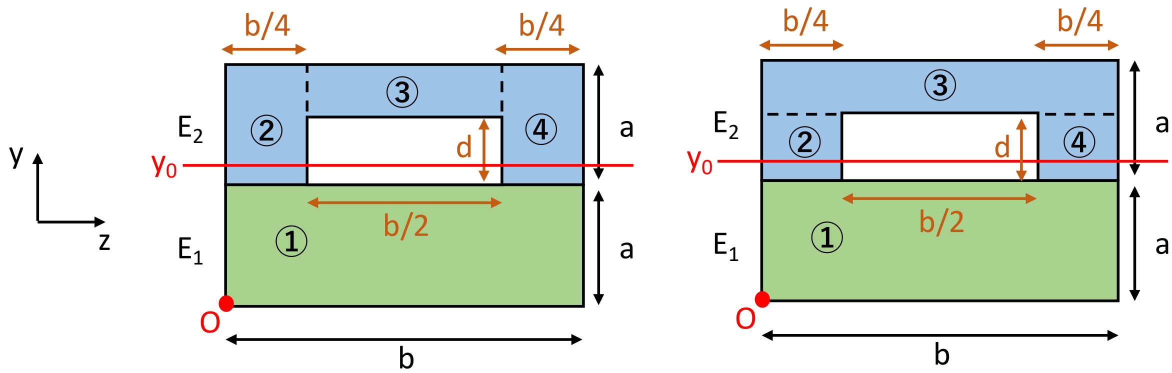
$$y_0 = \frac{E_1 b \frac{a^2}{2} + E_1 b 3a^2 - E_1 \frac{b}{2}(2ad + d^2)}{E_1 ab + 2E_1 ab - E_1 bd}$$

$$= \frac{\frac{a^2}{2} + 3a^2 - \frac{1}{2}(2ad + d^2)}{a + 2a - d}$$

$$y_0 = \frac{\frac{7a^2}{2} - ad - \frac{d^2}{2}}{3a - d}$$

If $d = 0$, we get $y_0 = \frac{7}{6}a > a$, putting the neutral axis in the blue material, which is reasonable since the blue material is more rigid.

Alternative Q1b: We could also have divided the beam in different ways. Here are 2 examples.



For the left example,

Zone 2:

$$A_2 = E_2 \int_{z=0}^{b/4} \int_{y=a}^{2a} dy\, dz = \frac{1}{2} E_1 ab$$

$$T_2 = E_2 \int_{z=0}^{b/4} \int_{y=a}^{2a} y dy\, dz = 2E_1 \frac{b}{4}\frac{1}{2} y^2 \Big|_a^{2a} = E_1 \frac{b}{4}(4a^2 - a^2) = \frac{3}{4} E_1 ba^2$$

Zone 4 (similar to zone 2): $A_4 = \frac{1}{2} E_1 ab$, $T_4 = \frac{3}{4} E_1 ba^2$

Zone 3:

$$A_3 = E_2 \int_{z=b/4}^{3b/4} dz \int_{y=a+d}^{2a} dy = 2E_1 \frac{b}{2}(a - d) = E_1 b(a - d)$$

$$T_3 = E_2 \int_{z=b/4}^{3b/4} dz \int_{y=a+d}^{2a} y dy = 2E_1 \frac{b}{2} \frac{y^2}{2} \Big|_{a+d}^{2a} = E_1 \frac{b}{2}[(2a)^2 - (a+d)^2]$$

$$= E_1 \frac{b}{2}(3a^2 - 2ad - d^2)$$

For the right example,

Zone 2:

$$A_2 = E_2 \int_{z=0}^{b/4} \int_{y=a}^{a+d} dy\, dz = \frac{1}{2} E_1 bd$$

$$T_2 = E_2 \int_{z=0}^{b/4} \int_{y=a}^{a+d} y dy\, dz = 2E_1 \frac{b}{4} \frac{1}{2} y^2 \bigg|_a^{a+d} = E_1 \frac{b}{4} [(a+d)^2 - a^2]$$
$$= \frac{1}{4} E_1 b(2ad + d^2)$$

Zone 4 (similar to zone 2): $A_4 = \frac{1}{2} E_1 bd, T_4 = \frac{1}{4} E_1 b(2ad + d^2)$

Zone 3:

$$A_3 = E_2 \int_{z=0}^{b} dz \int_{y=a+d}^{2a} dy = 2E_1 b(a-d)$$

$$T_3 = E_2 \int_{z=0}^{b} dz \int_{y=a+d}^{2a} y dy = 2E_1\, b \frac{y^2}{2} \bigg|_{a+d}^{2a} = E_1 b[(2a)^2 - (a+d)^2]$$
$$= E_1 b(3a^2 - 2ad - d^2)$$

For both alternative examples, Zone 1 is the same as explained above. The position of the neutral axis y_0 is given by:

$$y_0 = \frac{T_1 + T_2 + T_3 + T_4}{A_1 + A_2 + A_3 + A_4} = \frac{\frac{7a^2}{2} - ad - \frac{d^2}{2}}{3a - d}$$

Q1c)

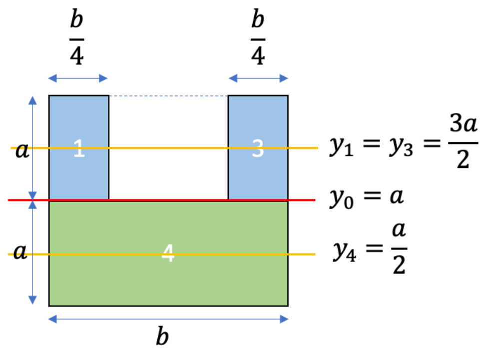
We have $y_0 = a$,

$$y_0 = \frac{\frac{7a^2}{2} - ad - \frac{d^2}{2}}{3a - d} = a$$

$$\frac{7}{2} a^2 - 3a^2 = \frac{d^2}{2}$$
$$\frac{a^2}{2} = \frac{d^2}{2}$$

$$d = a$$

Q1d)



$\langle EI \rangle = \sum_i E_i I_{zy_{0\,i}}$

Q1d Option 1: we sum areas 1,3 and 4. We separately compute I_{i,y_i} for areas 1, 3, 4 about their "local" neutral axis y_1, y_3, and y_4. Then, we get the I_{i,y_0} values for areas 1, 3, 4 about y_0, using "Steiner".

$$y_1 = y_3 = \frac{3}{2}a,\; I_{1,y_1} = \frac{1}{12}a^3\frac{b}{4} = I_{3,y_3}$$

$$A_1 = A_3 = a\frac{b}{4}$$

$$I_{1y_0} = I_{1y_1} + (y_1 - y_0)^2 A_1 = \frac{a^3 b}{48} + (\frac{3}{2}a - a)^2\frac{ab}{4} = \frac{1}{12}a^3 b =$$

$$I_{1y_0} = I_{3y_0} = \frac{1}{12}a^3 b$$

$$y_4 = \frac{a}{2}, \qquad I_{4y_4} = \frac{1}{12}a^3 b$$

$$I_{4,y_0} = I_{4y_4} + (y_4 - y_0)^2 A_4 = \frac{1}{12}ba^3 + (\frac{a}{2} - a)^2 ab$$

$$I_{4,y_0} = \frac{1}{3}a^3 b$$

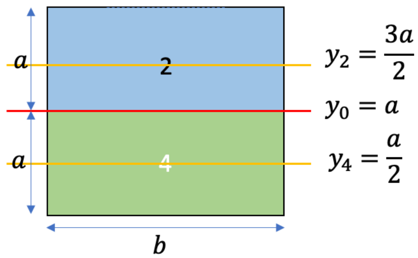
Thus

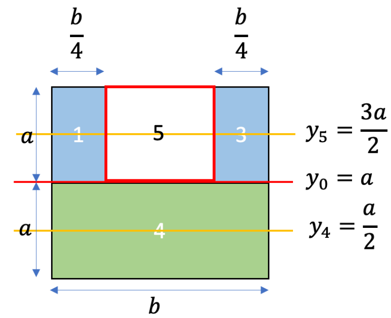
$$\langle EI \rangle = E_2 I_{1y_0} + E_2 I_{3y_0} + E_1 I_{4y_0}$$

$$\langle EI \rangle = 2E_1 * \frac{1}{12}a^3 b + 2E_1 * \frac{1}{12}a^3 b + E_1 * a^3\frac{b}{3}$$

$$\langle EI \rangle = \frac{2}{3}E_1 a^3 b$$

Alternative Q1d





option 2: we sum $E_i\ I_{i,y_i}$ for areas 2 and 4 and subtract $E_i\ I_{i,y_i}$ for area 5. Like before, we need to use Steiner to go from y_i to y_0

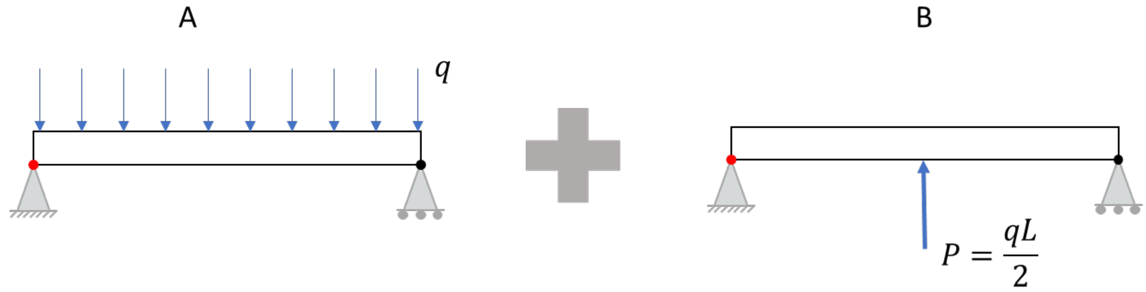
$$I_{top,y2} = I_{2,y_2} - I_{5y_5} = \frac{1}{12}ba^3 - \frac{1}{12}\frac{b}{2}a^3 = \frac{1}{24}a^3b$$

$$I_{top,y0} = \frac{1}{24}a^3b + \left(\frac{3}{2}a - a\right)^2 (ab - \frac{ab}{2}) = \frac{1}{6}a^3b$$

$$\langle EI\rangle = E_2(I_{top,y0}) + E_1 I_{4y_0} = 2E_1 * \frac{1}{6}a^3b + E_1 * a^3\frac{b}{3} = \frac{2}{3}E_1a^3b$$

Q1d Option 3: using integrals: one can also use the integral definition of $\langle EI\rangle$

Q1e) finding $w(x)$ using the Superposition method:



$$w(x) = w_A(x) + w_B(x)$$

From the tables we find $w_A(x)$ and $w_B(x)$

$$w_A(x) = \frac{-qx}{24\langle EI\rangle}(L^3 - 2Lx^2 + x^3), \qquad 0 \le x \le L$$

$$\begin{cases} w_{B1}(x) = +\frac{qL}{2}x\frac{1}{48\langle EI\rangle}(3L^2 - 4x^2) = \frac{qLx(3L^2-4x^2)}{96\langle EI\rangle}, & 0 \le x \le \frac{L}{2} \\ w_{B2}(x) = +\frac{qL}{2}(L-x)\frac{1}{48\langle EI\rangle}[3L^2 - 4(x-L)^2] = \frac{qL(L-x)[3L^2-4(x-L)^2]}{96\langle EI\rangle}, & \frac{L}{2} \le x \le L \end{cases}$$

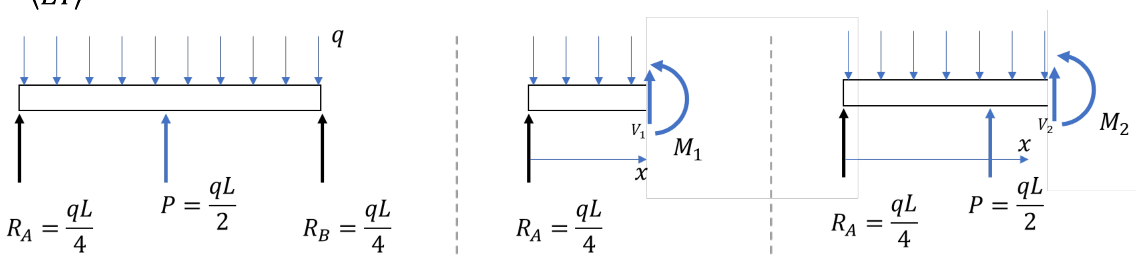
Combining:

$$w(x) = \begin{cases} w_A(x) + w_{B1}(x), & 0 \le x \le \frac{L}{2} \\ w_A(x) + w_{B2}(x), & \frac{L}{2} \le x \le L \end{cases}$$

$$= \begin{cases} \frac{1}{\langle EI\rangle}\left(-\frac{1}{24}qx^4 + \frac{1}{24}qLx^3 - \frac{1}{96}qL^3x\right) & 0 \le x \le \frac{L}{2} \\ \frac{1}{\langle EI\rangle}\left(-\frac{1}{24}qx^4 + \frac{3}{24}qLx^3 - \frac{1}{8}qL^2x^2 + \frac{5}{96}qL^3x - \frac{1}{96}qL^4\right) & \frac{L}{2} \le x \le L \end{cases}$$

Q1e) Alternative method: integrating $w(x)$ from $M(x)$, after finding M(x) by sections

$$\frac{d^2w}{dx^2} = \frac{M(x)}{\langle EI\rangle}$$



$$M_1(x) = -\frac{1}{2}q(L-x)^2 + \frac{qL}{4}(L-x) + \frac{qL}{2}(\frac{L}{2}-x) = -\frac{1}{2}qx^2 + \frac{1}{4}qLx, 0 \le x \le \frac{L}{2}$$
$$M_2(x) = -\frac{1}{2}q(L-x)^2 + \frac{qL}{4}(L-x) = -\frac{1}{2}qx^2 - \frac{1}{4}qL^2 + \frac{3}{4}qLx, \frac{L}{2} \le x \le L$$

Do a first integration to find $w'(x)$:

$$\frac{dw}{dx} = \begin{cases} \frac{-1}{\langle EI \rangle}(\frac{1}{6}qx^3 - \frac{1}{8}qLx^2 + C_1), & 0 \le x \le \frac{L}{2} \\ \frac{-1}{\langle EI \rangle}(\frac{1}{6}qx^3 - \frac{3}{8}qLx^2 + \frac{1}{4}qL^2x + C_2), & \frac{L}{2} \le x \le L \end{cases}$$

Do a 2nd integration to get $w(x)$:

$$w(x) = \begin{cases} \frac{-1}{\langle EI \rangle}(\frac{1}{24}qx^4 - \frac{1}{24}qLx^3 + C_1x + C_3), & 0 \le x \le \frac{L}{2} \\ \frac{-1}{\langle EI \rangle}(\frac{1}{24}qx^4 - \frac{3}{24}qLx^3 + \frac{1}{8}qL^2x^2 + C_2x + C_4), & \frac{L}{2} \le x \le L \end{cases}$$

Boundary condition and continuity conditions allow finding the four constants:

$$w(0) = 0$$
$$w(L) = 0$$
$$w(x)_{x=(\frac{L}{2})^+} = w(x)_{x=(\frac{L}{2})^-}$$
$$\left(\frac{dw}{dx}\right)_{x=(\frac{L}{2})^+} = \left(\frac{dw}{dx}\right)_{x=(\frac{L}{2})^-}$$

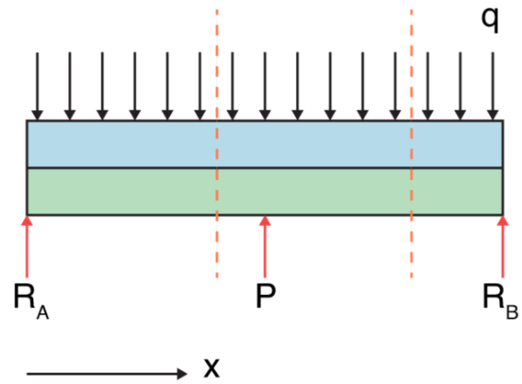
Solution for the four constants:

$$C_1 = \frac{1}{96}qL^3, \quad C_2 = \frac{-5}{96}qL^3, \quad C_3 = 0, \quad C_4 = \frac{1}{96}qL^4$$

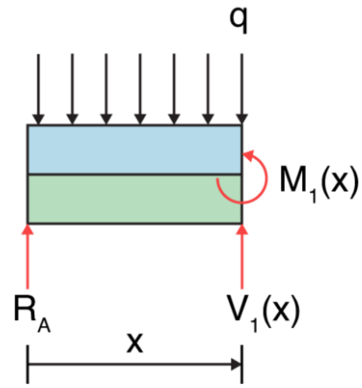
Overall, solution for the deflection $w(x)$ is:

$$w(x) = \begin{cases} \frac{1}{\langle EI \rangle}(-\frac{1}{24}qx^4 + \frac{1}{24}qLx^3 - \frac{1}{96}qL^3x), & 0 \le x \le \frac{L}{2} \\ \frac{1}{\langle EI \rangle}(-\frac{1}{24}qx^4 + \frac{3}{24}qLx^3 - \frac{1}{8}qL^2x^2 + \frac{5}{96}qL^3x - \frac{1}{96}qL^4), & \frac{L}{2} \le x \le L \end{cases}$$

Q1f) Method 1: by sections



We need to cut the beam twice. For the first cut, $0 \leq x \leq L/2$,



Sum of forces in y: $V_1(x) + R_{Ay} - q \cdot x = 0,$

$$V_1(x) = qx - \frac{qL}{4},$$

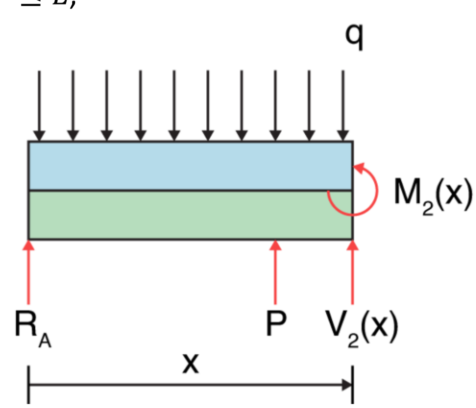
Sum of Moments at point A:

$$V_1(x) \cdot x + M_1(x) - q \cdot x \cdot \frac{x}{2} = 0,$$

Therefore, for $0 \leq x \leq L/2$,

$$M_1(x) = -\frac{qx^2}{2} + \frac{qxL}{4},$$

For the second cut, $L/2 \leq x \leq L$,



Sum of forces in *y*: $V_2(x) + R_{Ay} - q \cdot x + P = 0,$

$$V_2(x) = qx - \frac{qL}{4} - \frac{qL}{2},$$

Sum of Moments at point A:

$$V_2(x) \cdot x + M_2(x) + P \cdot \frac{L}{2} - q \cdot x \cdot \frac{x}{2} = 0,$$

Therefore, for $L/2 \le x \le L$,

$$M_2(x) = -\frac{qx^2}{2} + \frac{3qxL}{4} - \frac{qL^2}{4},$$

Q1f Method 2: by integrating the load to get $V(x)$, then integrating $V(x)$ to get $M(x)$

$q(x) = q$, for 0 ⩽ x ⩽ L, with a point load *P* at $x = L/2$

For $0 \le x \le L/2$,

$$V_1(x) = \int q(x) + C_1 = -q \cdot x + C_1$$

For $L/2 \le x \le L$,

$$V_2(x) = \int \mathrm{q(x)} + C_2 = -q \cdot x + C_2$$

Since

$V_1(0) = R_{Ay} = \frac{qL}{4},$

We have

$C_1 = \frac{qL}{4},$

The point load *P* give a step in $V(x)$, so

$$V_2(L/2) = V_1(L/2) + P$$

Hence

$C_2 = \frac{\mathrm{3qL}}{4},$

Therefore,

$$V_1(x) = -q \cdot x + \frac{qL}{4}$$

$$V_2(x) = -q \cdot x + \frac{3qL}{4}$$

We now integrate $V(x)$ to find $M(x)$

For $0 \le x \le L/2$,

$$M_1(x) = \int V_1(x) + C_3 = -\frac{qx^2}{2} + \frac{qxL}{4} + C_3$$

For $L/2 \le x \le L$,

$$M_2(x) = \int V_2(x) + C_4 = -\frac{qx^2}{2} + \frac{3qxL}{4} + C_4$$

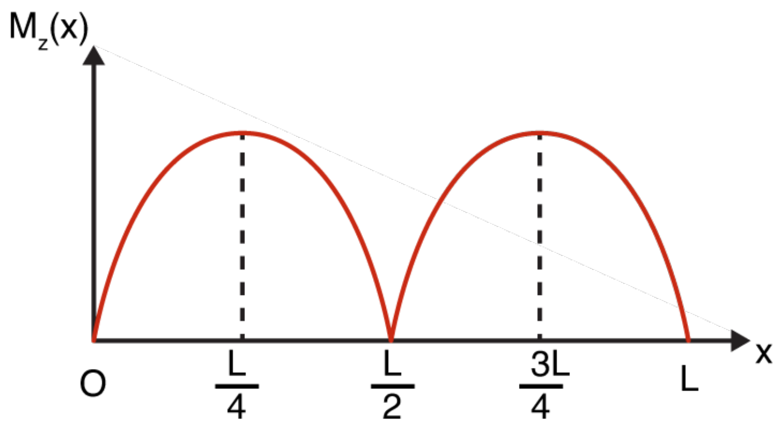
Since $M_1(0) = 0$, and $M_2(L) = 0$, we have:

$$C_3 = 0,$$
$$C_4 = -\frac{qL^2}{4}$$

Thus, we obtain (as above):

$$M_1(x) = -\frac{qx^2}{2} + \frac{qxL}{4},$$
$$M_2(x) = -\frac{qx^2}{2} + \frac{3qxL}{4} - \frac{qL^2}{4}$$

Max of $M_z(x)$



To find the maximum, we compute M'(x)=0, and find $x = L/4$ and $x = 3L/4$

For $0 \leq x \leq L/2$,

$$M_1(x)_{max} = M_1\left(\frac{L}{4}\right) = \frac{qL^2}{32},$$

For $L/2 \leq x \leq L$,

$$M_2(x)_{max} = M_2\left(\frac{3L}{4}\right) = \frac{qL^2}{32},$$

Q1g)

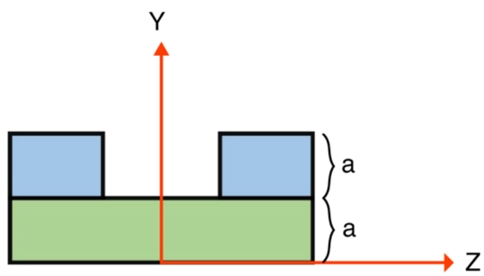
We know that $y_0 = a$, and we also have:

$$M_1\left(\frac{L}{4}\right) = \frac{qL^2}{32}$$

$$\langle EI\rangle = \frac{2E_1a^3b}{3}$$

We know:

$$\sigma_x(y) = \frac{-E(y)\cdot M_z(x)}{\langle EI\rangle}\cdot(y-y_0)$$



Therefore, for the blue part ($a \leqslant y \leqslant 2a$), we have:

$$\sigma_x(y) = \frac{-2E_1\cdot M_1\left(\frac{L}{4}\right)}{\langle EI\rangle}\cdot(y-a)$$

$$\sigma_x(y) = \frac{-2E_1\cdot\frac{qL^2}{32}}{\frac{2E_1a^3b}{3}}\cdot(y-a) = -\frac{3qL^2}{32a^3b}\cdot(y-a)$$

For the green part at the bottom ($0 \leqslant y \leqslant a$), we have:

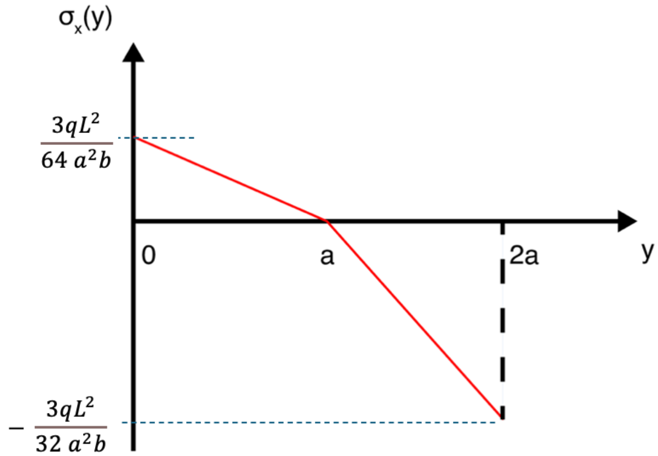
$$\sigma_x(y) = \frac{-E_1\cdot M_1\left(\frac{L}{4}\right)}{EI}\cdot(y-a)$$

$$\sigma_x(y) = \frac{-E_1\cdot\frac{qL^2}{32}}{\frac{2E_1a^3b}{3}}\cdot(y-a) = -\frac{3qL^2}{64a^3b}\cdot(y-a)$$

The maximum $|\sigma_x(y)|$ occurs at $y = 2a$, located on the top of the blue part,

$$\sigma_x(y=2a) = -\frac{3qL^2}{32a^2b} \quad < 0$$

Which means the top of the beam is in compression.



Approximate shape of the beam: we can see that at L/4 the top is in compression and the bottom is in tension.